Canadian Journal of Pure and Applied Sciences
Vol. 12, No. 1, pp. 4465-4473, Feb 2018
Online ISSN: 1920-3853; Print ISSN: 1715-9997
Available online at www.cjpas.net

NEW CONDITIONS FOR THE NEWTON-KANTOROVICH APPROXIMATIONS TO NONLINEAR SINGULAR INTEGRAL EQUATIONS WITH CARLEMAN SHIFT OF FINITE ORDER

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ABSTRACT

The paper is concerned with the applicability of some new conditions for the convergence of Newton–kantorovich approximations to solution of a class of nonlinear singular integral equations with Carleman shift, forming the finite group of iterations preserving orientation**,** of Uryson type**.** The results are illustrated in generalized Holder space.

AMS: Subject classification (2010) 45G50, 47H10, 47H30, 26A16
Keywords and phrases: Newton–Kantorovich approximations**,** nonlinear singular integral equations of uryson type, noether operator, carleman shift, generalized holder space.

INTRODUCTION

The theory of integral equations is one of the most important branches of mathematical analysis, mainly result of its importance in boundary value problems in the theory of partial differential equations. The theory of integral equations is contacted with many different areas of mathematics; foremost among these are differential equations, theory of analytic functions and operator theory.

The theory of linear singular integral equations (SIE) and non-linear singular integral equations (NSIE) have been developed significant importance during the last years as many engineering problems of applied mechanics and applied mathematics and in many problems of mathematical physics, such as the theory of elasticity, hydrodynamics, quantum mechanics, fluid mechanics and others. The theory of approximation methods and its applications for the solution of SIE and NSIE has been developed by many authors (Amer, 1996; Dardery, 2011, 2017; Dardery and Allan, 2011; Jinyuan, 2000; Kantorovich and Akilov, 1982; Ladopoulous and Zisis, 1996; Zabrejko and Nguen, 1987). The classical and more recent results on the solvability of NSIE should be generalized to corresponding equations with shift see Wolfersdorf (1985). The successful development of the theory of SIE and NSIE naturally stimulated the study of singular integral equations with shift (SIES) and nonlinear singular integral equations with shift (NSIES), (Amer, 2001; Amer and Dardery, 2005; Dardery, 2011, 2014; Gakhov, 1966; Kravchenko and Litvinchuk, 1994; Litvinchuk, 1977; Nguyen, 1989).

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The theory of SIES and NSIES are an important part of integral equations because of its recent applications in many fields of physics and engineering, (Baturev, 1996; Kravchenko, 1995; Kravchenko, 1994). The Weiner-Hope equations are a natural apparatus for the solution of problems of synthesis of signals for linear systems with continuous time and stationary parameters. If the problem of synthesis is not stationary, then the Weiner-Hope method is not applicable and the problem is reduced to singular integral equation (Baturev, 1996; Cooper, 1971).

Exact and approximate solutions of such equations attracted many mathematicians. The known results concerning on a criterion of Noetherity and index formula for singular integral functional operator with (Carleman or non-Carleman) shift (SIOS) preserving or changing the orientation are investigated in the case of continuous coefficients in the recent monograph by Kravechenko and Litvinchuk (1994). The Noether theory of SIOS is developed for a closed and open contour (Amer and Dardery, 2004, 2009; Dardery, 2011; Guseinov, and Mukhtarov, 1980; Khusnutdinov, 1989; Kravchenko *et al*., 1995; Kravchenko and Litvinchuk, 1994; Litvinchuk, 1977). In the present paper, some new conditions for the convergence of Newton-Kantorovich approximations have been applied to solution of the following NSIES of Uryson type:

$$(Tu)(t)=\sum_{i=0}^{m-1}\left(a_i(t)u\,(\alpha_i(t))+\lambda\frac{b_i(t)}{\pi i}\int_{\Gamma}\frac{\Psi(\alpha_i(t),\tau,u(\tau))}{\tau-\alpha_i(t)}d\tau\right)=0,\ t\in\Gamma \quad (1.1)$$

in generalized Holder space $H_{\Gamma}(\omega)$ where Γ be a simple smooth closed Lyapunov contour, which divides the plane of the complex variable Z into two domains, the interior domain D^{+} and the exterior domain D^{-}, $\alpha(t)$ homeomorphically maps Γ into itself with preservation orientation and satisfies the Carleman condition:

$$\alpha_m(t)=t,\ \alpha_i(t)\neq t,\ 1\le i\le m-1, \quad (1.2)$$

where

$$\alpha_i(t)=\alpha[\alpha_{i-1}(t)],\alpha_0(t)=t,$$

and $m\ge 2$. Assume that $\alpha'(t)$ satisfies the Holder condition. Moreover $\Psi:\Gamma\times\Gamma\times R\to R$ is a Caratheodory function (i.e. function which is continuous in the last variable and measurable in the other variables), Also, we suppose that the derivative of the Caratheodory function $\Psi(t,\tau,u)$ with respect to the last variable exists and is also a caratheodory function. The coefficients $a_i(t),b_i(t),i=0,1,...,m-1$ belong to the generalized Holder space $H_{\Gamma}(\omega)$ and $\lambda\in(-\infty,\infty)$, is a numerical parameter; the function $u(t)$ is an unknown function. The usefulness of the following study consists in reducing the (hard) problem of finding zero of a nonlinear operator in a Banach space to the (possible simpler) problem of finding zero of a scalar function.

Our problem has been studied when

$$a_i(t)=0,i=0,1,...,m-1$$

by applicability of Banach fixed-point theorem in (Amer and Dardery, 2004), also it has been studied where Γ is a real segment in usual Holder space in (Nguyen, 1988). The special case has been studied, with first order shift, in (Dardery, 2011). The special case of our problem has been studied as nonlinear integral equation, without shift, in the chebyshev space C, the lebesgue space $L_p(1\le p\le\infty)$, and orlicz space L_M (De Pascale and Zabreiko, 1998).

2- Formulation of the problem:

Let X and Y be two Banach spaces, $B(u_0,R)=\{u:u\in X,\ \|u-u_0\|<R\}$ the closed ball centered at $u_0\in X$ with radius $R>0$, and $F:B(u_0,R)\to Y$ is nonlinear operator. The Newton-Kantorovich method is one of the basic tools for finding approximate solutions of the operator

$$F(u)=0 \quad (2.1)$$

In the corresponding iterative scheme

$$u_{n+1}=u_n-F'(u_n)^{-1}F(u_n),\qquad (n=0,1,2,...) \quad (2.2)$$

One has to require in particular that the Frechet derivative of F at all points u_n exists and is invertible in the Banach space $C(X,Y)$ of all bonded linear operators from X into Y. The non-negative numbers

$$a=\left\|F'(u_0)^{-1}F(u_0)\right\| \quad (2.3)$$

$$b=\left\|F'(u_0)^{-1}\right\| \quad (2.4)$$

Will be of particular interest to us in what follows.

We suppose that the Frechet derivative $F'(u)$ of F satisfies at each point of $B(u_0,R)$ a condition of the form

$$\left\|F'(u_1)-F'(u_2)\right\|\le\mu(\|u_1-u_2\|),\quad u_1,u_2\in B(u_0,R) \quad (2.5)$$

Where $\mu:[0,\infty)\to[0,\infty)$ is monotonically increasing with

$$\lim_{r\to 0}\mu(r)=0,\qquad 0<r<R \quad (2.6)$$

Moreover, we assume that there is another monotonically increasing function $\theta:[0,\infty)\to[0,\infty)$ such that $0\le\theta(r)\le\mu(r),\ (0\le r\le R)$, and

$$\left\|F'(u)^{-1}\right\|\le\frac{b}{1-b\theta(r)}\ ,(u\in B(u_0,r))\,. \quad (2.7)$$

We define three scalar functions on $[0,R]$ by

$$\tilde{\mu}(r)=\sup\{\mu(u)+\theta(v):u+v=r\}, \quad (2.8)$$

$$\phi(r)=\frac{a}{b}+\int_0^r\mu(t)dt-\frac{r}{b}\ ,\ (0\le r\le R)\,, \quad (2.9)$$

and

$$\tilde{\phi}(r)=\frac{a}{b}+\int_0^r\tilde{\mu}(t)dt-\frac{r}{b}\ ,(0\le r\le R)\,. \quad (2.10)$$

Theorem 2.1 (De Pascale and Zabreiko, 1998). Suppose that the function (2.9) has a unique zero $r_0\in[0,R]$ and that $\phi(R)\le 0$. Then equation (2.1) has a solution $x_*\in B(x_0,r_0)$ this is unique in the ball $B(u_0,R)$

Lemma 2.1 (De Pascale and Zabreiko, 1998). Suppose that the function (2.10) has a unique zero $q_*\in[0,R]$ and that $\tilde{\phi}(R) \le 0$. Then the scalar sequence $(r_n)_{n\in N}$ defined by

$$r_0 = 0, \quad r_{n+1} = r_n + \frac{b\tilde{\phi}(r_n)}{1 - b\theta(r_n)} \quad (n = 0,1,2,...) \tag{2.11}$$

Converges monotonically to q_*

Theorem 2.2 (De Pascale and Zabreiko, 1998). Under the hypotheses of Lemma 2.1 the approximations (2.2) are defined for all n belong to the ball are converging to a solution of (2.1) and satisfy the estimates

$$\| u_{n+1} - u_n \| \le r_{n+1} - r_n \ , \ (n = 0,1,2,\ldots), \tag{2.12}$$

and

$$\| u_* - u_n \| \le q_* - r_n \ , \ (n = 0,1,2,\ldots) \tag{2.13}$$

Theorem 2.3 (De Pascale and Zabreiko, 1998). Suppose that the sequence $(r_n)_n$ given by (2.11) converges to some limit $r_\infty(a)$. Then the approximations (2.2) are defined for all n belong to the ball $B(u_0, r_\infty(a))$, and satisfy the estimates (2.12) and (2.13).

We remark that the usefulness of Theorem 2.1 consists in reducing the (hard) problem of finding zero of a nonlinear operator in a Banach space to the (possible simpler) problem of finding zero of a scalar function.

3- Some notations and auxiliary results.

In this section, we introduce some notations and auxiliary results, which will be used in the sequel.

Definition 3.1 (Guseinov and Mukhtarov, 1980). We denote by Φ the class of all functions $\omega(\delta)$, defined on $(0, l]$, where l is the length of the curve Γ, which satisfies the following conditions:

1. $\omega(\delta)$ is a modulus of continuity,
2. $\sup\limits_{\delta>0} \dfrac{1}{\omega(\delta)} \int_0^\delta \dfrac{\omega(s)}{s} ds = I_\omega < \infty$,
3. $\sup\limits_{\delta>0} \dfrac{\delta}{\omega(\delta)} \int_\delta^l \dfrac{\omega(s)}{s^2} ds = J_\omega < \infty.$

Definition 3.2 (Guseinov and Mukhtarov, 1980; Mikhlin and Prossdorf, 1986). The generalized Holder space $H_\Gamma(\omega)$ is the set of all continuous function $u(t)$ such that

$$H_\Gamma^\omega(u) = \sup_{t_1, t_2 \in \Gamma} \frac{|u(t_1) - u(t_2)|}{\omega(|t_1 - t_2|)} < \infty \cdot$$

For $u \in H_\Gamma(\omega)$ we define the norm:

$$\| u \|_{H_\Gamma} = \| u \|_{c(\Gamma)} + H_\Gamma^\omega(u),$$

where

$$\| u \|_{c(\Gamma)} = \max_{t\in\Gamma} |u(t)|.$$

Using the notations

$$(\Lambda_\Psi u)(t) = \sum_{i=0}^{m-1} \left(\frac{\lambda b_i(t)}{\pi i} \int_\Gamma \frac{\Psi(\alpha_i(t), \tau, u(\tau))}{\tau - \alpha_i(t)} d\tau \right), \ t \in \Gamma \tag{3.1}$$

$$(L_\Psi u)h(t) = \sum_{i=0}^{m-1} \left(\frac{\lambda b_i(t)}{\pi i} \int_\Gamma \frac{l(\alpha_i(t), \tau, u(\tau))}{\tau - \alpha_i(t)} h(\tau) d\tau \right), \ h(t) \in H_\Gamma(\omega) \tag{3.2}$$

$$l(t, \tau, u) = \frac{\partial \Psi(t, \tau, u)}{\partial u} \tag{3.3}$$

and

$$(S u)(t) = \frac{1}{\pi i} \int_\Gamma \frac{u(\tau)}{\tau - t} \tag{3.4}$$

for singular integral operator,

$$P_\pm = \frac{1}{2}(I \pm S), \quad S^2 = I \tag{3.5}$$

to which we associate the projection operators where I is the identity operator on $H_\Gamma(\omega)$

$$(Wu)(t) = u(\alpha(t)),$$

for the Carleman shift operator, and the operators B_1, B_2 are defined by

$$(B_1 u)(t) = \sum_{i=0}^{m-1} a_i(t) W^i u(t), (B_2 u)(t) = \sum_{i=0}^{m-1} b_i(t) W^i u(t) \cdot \tag{3.6}$$

where

$$(W^i u)(t) = u(\alpha_i(t)), \qquad i = 0,1,..., m-1.$$

Lemma 3.1 (Amer, 2001**).** The singular operator S is a bounded operator on the space $H_\Gamma(\omega)$ and satisfies the inequality

$$\|Su\|_{H_\Gamma} \le \rho_0 \|u\|_{H_\Gamma}, \tag{3.7}$$

where ρ_0 is a constant defined as follows :

$$\rho_0 = c_1 \int_0^\delta \frac{\omega(\xi)}{\xi} d\xi + c_1 + c_2 \tilde{c},$$

where $\tilde{c}$ is a positive constant.

Theorem 3.1 (Amer and Dardery, 2004). The shift operators $B_i\,; i=1,2$ are bounded operators on the generalized Holder space $H_\Gamma(\omega)$ and satisfy the inequality

$$\|(B_i u)(t)\|_{H_\Gamma} \le \Theta_i \|u\|_{H_\Gamma}\,;\ i=1,2 \qquad (3.8)$$

where

$$\Theta_i = \max\{ M_{B_i}, L_{B_i} \},$$

$$M_{B_1} = \sum_{i=0}^{m-1} \|a_i(t)\|_c\,,\quad M_{B_2} = \sum_{i=0}^{m-1} \|b_i(t)\|_c$$

and

$$L_{B_1} = \sum_{i=0}^{m-1} H_\Gamma^\omega(a_i),\quad L_{B_2} = \sum_{i=0}^{m-1} H_\Gamma^\omega(b_i),$$

Now, we study the singular integral operator Λ_Ψ defined by the equality (3.1) where the function $\Psi=\Psi(t,\tau,u):\Gamma\times\Gamma\times R\to R$ satisfies the following condition

$$\left|\Psi_{u^{(i)}}(t_1,\tau_1,u_1)-\Psi_{u^{(i)}}(t_2,\tau_2,u_2)\right| \le A_1^i\,\omega^*(|t_1-t_2|)+ A_2^i\omega(|\tau_1-\tau_2|)\ +\xi_i(|u_1-u_2|);\ \ i=0,1 \qquad (3.9)$$

for $\omega(\delta),\omega^*(\delta)\in\Phi$, we have

$$\omega^*(\delta)\ln(l/\delta)\le A_3\omega(\delta), \qquad (3.10)$$

$\xi_i:[0,\infty)\to[0,\infty)$ is monotonically increasing function with

$$\lim_{r\to o}\xi_i(r)=0;\quad 0\le r\le R, \qquad (3.11)$$

where A_1^i, A_2^i and A_3 are positive constants.

Lemma 3.2. If the function $\Psi(t,\tau,u)$ satisfies the conditions (3.9)-(3.11), then the operator Λ_Ψ defined by (3.1) is bounded on $H_\Gamma(\omega)$ and satisfy the inequality

$$\|\Lambda_\Psi u(t)\|_{H_\Gamma} \le \Theta_2\left\{\wedge_1+\wedge_2+\rho_0\|\Psi(t,t,u(t)\|_{H_\Gamma}\right\}. \qquad (3.12)$$

Where $\wedge_1,\wedge_2$ are defined constant depend on constants A_1^i, A_2^i and A_3 (Dardery, 2011).

In the following Theorem, the function $m=m(t,\tau)$ should carry the following quite restrictive conditions:

1. $\sup\limits_{0<\delta<l} \dfrac{\omega_m(\delta,0)\ln(l(\delta))}{\omega(\delta)} = I_1<\infty$
2. $\sup\limits_{0<\delta<l} \dfrac{1}{\omega(\delta)}\displaystyle\int_0^\delta \frac{\omega_m(0,\xi)}{\xi}d\xi = I_2<\infty$
3. $\sup\limits_{0<\delta<l} \dfrac{\delta}{\omega(\delta)}\displaystyle\int_0^\delta \frac{\omega_m(0,\xi)}{\xi^2}d\xi = I_3<\infty$

Theorem 3.2. The nonlinear singular operator L_Ψ defined by inequality (3.2) and (3.3) is a bounded operator on the generalized Holder space $H_\Gamma(\omega)$ and satisfy the inequality

$$\|L_\Psi(u)h\|_{H_\Gamma} \le \gamma\|h\|_{H_\Gamma} \qquad (3.13)$$

where

$$\gamma=\theta_2.\max\{I_4+\tilde{I}_4, I_5+\tilde{I}_5\}.$$

and $I_4,\tilde{I}_4,I_5,\tilde{I}_5$ are defined constant depend on constants I_1,I_2,I_3,I_ω and J_ω , (Dardery, 2011).

Lemma 3.3. Let the function $\Psi(t,s,u)$ satisfies the conditions (3.9)-(3.11), then the operator $T(u)$ has Frechet differentiable at every fixed point $u\in H_\Gamma(\omega)$ and its derivative given by

$$(T'u)h(t)=\sum_{i=0}^{m-1}\left(a_i(t)h(\alpha_i(t))+\lambda\frac{b_i(t)}{\pi i}\int_\Gamma \frac{l\,(\alpha_i(t),\tau,u(\tau))}{\tau-\alpha_i(t)}h(\tau)d\tau\right),\ t\in\Gamma \qquad (3.14)$$

Satisfies the following condition

$$\|T'(u_1)-T'(u_2)\|_{H_\Gamma} \le \mu(\|u_1-u_2\|_{H_\Gamma}),\ u_1,u_2\in B(u_0,R),\ 0<r<R \qquad (3.15a)$$

where

$$\lim_{r\to 0}\mu(r)=0,\quad (0<r<R) \qquad (3.15b)$$

Proof.

Let $u_0(t)$ be a fixed element in the space $H_\Gamma(\omega)$ and $h(t)$ be an arbitrary element in $H_\Gamma(\omega)$. Now we consider

$$T(u_0+h)-T(u_0)=T'(u_0)h+\eta(u_0,h) \qquad (3.16)$$

then we have the well-known formula:

$$\Psi\left(t,\tau,u_0(\tau)+h(\tau)\right)-\Psi(t,\tau,u_0(\tau))=l\left(t,\tau,u_0(\tau)\right)h(\tau)+\eta^*\left(u_0+h\right) \quad (3.17)$$

where

$$\eta^*\left(u_0,h\right)=\int_0^1 (1-\vartheta)\,\Psi_{uu}\left(t,\tau,u_0(\tau)+\vartheta h(\tau)\right)h^2(\tau)d\vartheta, \quad (3.18)$$

where $0\le\vartheta\le 1$, from (3.16) and (3.17), we get

$$(T'u_0)h(t)=\sum_{i=0}^{m-1}\left(a_i(t)h(\alpha_i(t))+\lambda\frac{b_i(t)}{\pi i}\int_\Gamma\frac{l\left(\alpha_i(t),\tau,u_0(\tau)\right)}{\tau-\alpha_i(t)}h(\tau)d\tau\right),\ t\in\Gamma$$

$$\eta(u_0,h)=\lambda\sum_{i=0}^{m-1}\frac{b_i(t)}{\pi i}\int_\Gamma\frac{\int_0^1(1-\vartheta)\,\Psi_{uu}\left(\alpha_i(t),\tau,u_0(\tau)+\vartheta h(\tau)\right)h^2(\tau)d\vartheta}{\tau-\alpha_i(t)}\,d\tau$$

Moreover,

$$\|T(u_0+h)-T(u_0)-T'(u_0)h\|_{H_\Gamma}\le\|B_1h\|_{H_\Gamma}+$$
$$+\left\|\lambda B_2\int_\Gamma\frac{\left(\Psi(t,\tau,u_0(\tau)+h(\tau))-\Psi(t,\tau,u_0(\tau))-l(t,\tau,u_0(\tau))h(\tau)\right)}{\tau-t}d\tau\right\|_{H_\Gamma}$$
$$\left(\|h\|_{H_\Gamma}\right)$$

Hence

$$\|\eta(u_0,h)\|_{H_\Gamma}\to 0\ \ as\ \|h\|_{H_\Gamma}\to 0$$

Consequently, the operator $T(u)$ has Frechet differentiable in the space $H_\Gamma(\omega)$ and its derivative given by formula (3.14). Moreover, by inequalities (3.7)-(3.9) we get

$$\|T'(u_1)h-T'(u_2)h\|_{H_\Gamma}\le\left\|\lambda B_2\int_\Gamma\frac{\left(l(t,\tau,u_1(\tau))-l(t,\tau,u_2(\tau))\right)h(\tau)}{\tau-t}d\tau\right\|_{H_\Gamma}$$
$$\le|\lambda|\Theta_2\rho_0\xi_1\left(\|u_1-u_2\|\right)\|h\|_{H_\Gamma}$$

Hence,

$$\|T'(u_1)-T'(u_2)\|_{H_\Gamma}\le\mu(\|u_1-u_2\|_{H_\Gamma}),\ u_1,u_2\in B(u_0,R),\ 0<r<R$$

where

$$\lim_{r\to 0}\mu(r)=0,\quad (0<r<R)$$

Theorem 3.3. If the operators B_i and $L_\Psi(u)$ satisfy the inequalities (3.8), (3.13) respectively, then the nonlinear singular operator $T'(u)$ is a bounded operator on the generalized Holder space $H_\Gamma(\omega)$ and satisfies the inequality

$$\|T'(u)h\|_{H_\Gamma}\le(\Theta_1+\gamma)\|h\|_{H_\Gamma}$$

Where Θ_1,γ are defined constants, (Dardery, 2011).

4- Noether property and index formula for SIOS:

To study the Noether condition for the operator $T'u$, we reduce this operator to the following form

$$(T'u)h(t)=\sum_{i=0}^{m-1}\left(a_i(t)W^i+c_i(t)W^iS\right)h(t)+$$
$$+(Nh)(t)=g(t),\ \ t\in\Gamma\ ,m\ge 2 \quad (4.1)$$

Where

$$c_i(t)=\lambda b_i(t)l(\alpha_i(t),t,u(t)),$$
$$(Nh)(t)=\int_\Gamma R(t,\tau)h(\tau)d\tau$$

$$R(t,\tau)=\lambda\frac{b_i(t)}{\pi i}\int_\Gamma\frac{l\left(\alpha_i(t),\tau,u(\tau)\right)-l\left(\alpha_i(t),t,u(t)\right)}{\tau-\alpha_i(t)}h(\tau)d\tau,$$

reduce this operator to the form

$$\left(T'_{AB}\,h\right)(t)=\left(AP_++BP_-\right)h(t)=J(t), \quad (4.2)$$

where

$$A=\sum_{i=0}^{m-1}x_i(t)\,W_i,\qquad B=\sum_{i=0}^{m-1}y_i(t)\,W_i,$$

$$x_i(t)=a_i(t)+c_i(t),\ \ i=0,...,m-1,$$

$$y_i(t)=a_i(t)-c_i(t),\ \ i=0,...,m-1.$$

$$J(t)=g(t)-(Nh)(t),\ \ i=0,...,m-1.$$

From the theory of singular integral operators with shift, (Kravchenko and Litvinchuk, 1994), the Noether condition for the operator $T'u$ is given by:

$$\Delta_1(t)=\begin{bmatrix} x_0(t) & x_1(t) & \cdots & x_{m-1}(t)\\ x_{m-1}(\alpha(t)) & x_0(\alpha(t)) & \cdots & x_{m-2}(\alpha(t))\\ \vdots & \vdots & \cdots & \vdots\\ x_1(\alpha_{m-1}(t)) & x_2(\alpha_{m-1}(t)) & \cdots & x_0(\alpha_{m-1}(t))\end{bmatrix}\ne 0 \quad (4.3)$$

$$\Delta_2(t)=\begin{bmatrix} y_0(t) & y_1(t) & \cdots & y_{m-1}(t) \\ y_{m-1}(\alpha(t)) & y_0(\alpha(t)) & \cdots & y_{m-2}(\alpha(t)) \\ \vdots & \vdots & \cdots & \vdots \\ y_1(\alpha_{m-1}(t)) & y_2(\alpha_{m-1}(t)) & \cdots & y_0(\alpha_{m-1}(t)) \end{bmatrix}\neq 0, \tag{4.4}$$

hence the following theorem is valid

Theorem 4.1 (Kravchenko and Litvinchuk, 1994)
Consider the operator

$$T'_{AB}=AP_+ + BP_-,$$

where

$$A=\sum_{i=0}^{m-1} x_i(t)\,W^i, \qquad B=\sum_{i=0}^{m-1} y_i(t)\,W^i,$$

are orientation-preserving Carleman shift operators of order $m\geq 2$, $W^m=I$ and the matrix functions $x_i, y_i \in C^{n\times n}(\Gamma)$. Then T'_{AB} is Noetherian if and only if the condition (4.3) and (4.4) are satisfied and the index formula of the Noetherian operator T'_{AB} is given by:

$$ind\,T'_{AB}=\frac{1}{2\pi m}\left\{\arg\frac{\Delta_2(t)}{\Delta_1(t)}\right\}_\Gamma . \tag{4.5}$$

5- Solution of linear singular integral equation with shift:

Now, we show that the linear singular integral equation with shift (4.1) has a unique solution for every $g\in H_\Gamma(\omega)$. Apply the Operato

$$W, W^2, ..., W^{m-1}, S, WS, ..., W^{m-1}S$$

Successively to both sides of the equation (4.1), hence, we obtain the following system:

$$a_{m-1}(\alpha(t))h(t)+\sum_{i=0}^{m-2}a_i(\alpha(t))\left(W^{i+1}h\right)(t)+c_{m-1}(\alpha(t))(Sh)(t)+ \\ +\sum_{i=0}^{m-2}c_i(\alpha(t))\left(W^{i+1}Sh\right)(t)+(WNh)(t)=(Wg)(t) \tag{5.1}$$

$$a_{m-2}(\alpha_2(t))h(t)+a_{m-1}(\alpha_2(t))(Wh)(t)+\sum_{i=0}^{m-3}a_i(\alpha_2(t))\left(W^{i+2}h\right)(t)+ \\ +c_{m-2}(\alpha_2(t))(Sh)(t)+c_{m-1}(\alpha_2(t))(WSh)(t)+ \\ +\sum_{i=0}^{m-3}c_i(\alpha_2(t))\left(W^{i+2}Sh\right)(t)+\left(W^2Nh\right)(t)=\left(W^2g\right)(t) \tag{5.2}$$

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$$a_0(\alpha_{m-1}(t))\left(W^{m-1}h\right)(t)+\sum_{i=0}^{m-2}a_{i+1}(\alpha_{m-1}(t))\left(W^ih\right)(t)+ \\ +c_0(\alpha_{m-1}(t))\left(W^{m-1}h\right)(t)+\sum_{i=0}^{m-2}c_{i+1}(\alpha_{m-1}(t))\left(W^iSh\right)(t)+ \\ +\left(W^{m-1}Nh\right)(t)=\left(W^{m-1}g\right)(t) \tag{5.3}$$

$$\sum_{i=0}^{m-1}\left(c_i(t)\left(W^ih\right)(t)\right)+\sum_{i=0}^{m-1}a_i(t)\left(W^iS\right)h(t)+(N_1h)(t)=(Sg)(t), \tag{5.4}$$

$$c_{m-1}(\alpha(t))h(t)+\sum_{i=0}^{m-2}c_i(\alpha(t))\left(W^{i+1}h\right)(t)+a_{m-1}(\alpha(t))(Sh)(t)+ \\ +\sum_{i=0}^{m-2}a_i(t)\left(W^{i+1}Sh\right)(t)+(N_2h)(t)=(WSg)(t) \tag{5.5}$$

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$$\sum_{i=0}^{m-2}c_{i+1}(\alpha_{m-1}(t))\left(W^ih\right)(t)+c_0(\alpha_{m-1}(t))\left(W^{m-1}h\right)(t)+\sum_{i=0}^{m-2}a_{i+1}(\alpha_{m-1}(t))\left(W^iSh\right)(t)+ \\ +a_0(\alpha_{m-1}(t))\left(W^{m-1}Sh\right)(t)+(N_3h)(t)=\left(W^{m-1}Sg\right)(t) \tag{5.6}$$

Where

$$N_1=S\left(\sum_{i=0}^{m-1}a_i(t)W^i+\sum_{i=0}^{m-1}c_i(t)W^iS+N\right)-\left(\sum_{i=0}^{m-1}c_i(t)W^i+\sum_{i=0}^{m-1}a_i(t)W^iS\right)$$

$$N_2=WN_1$$

.
.
.

$$N_m=W^{m-1}S\left(\sum_{i=0}^{m-1}a_i(t)W^i+\sum_{i=0}^{m-1}c_i(t)W^iS+N\right)- \\ -W^{m-1}\left(c_0(t)W^{m-1}+\sum_{i=0}^{m-2}c_{i+1}(t)W^i+a_0(t)W^{m-1}S+\sum_{i=0}^{m-2}a_{i+1}(t)W^iS\right)$$

No solutions are lost when the operators $W, W^2, ..., W^{m-1}, S, WS, ..., W^{m-1}S$ are applied to equation (4.1), hence all solutions of (4.1) are solutions of the system (5.1)-(5.6) and conversely.

Let E be the closed subspace defined by

$$E = \{(h, Wh, ..., W^{m-1}h, Sh, WSh, ..., W^{m-1}Sh), \quad h \in H_\Gamma(\omega)\}$$

and let $\overline{\Omega}$ be the linear operator from E to $H_\Gamma(\omega)$ defined by

$$\overline{\Omega}H(t) = \Omega(t)H(t), \tag{5.7}$$

where

$$H = \begin{bmatrix} h & Wh & ... & W^{m-1}h & Sh & WSh & ... & W^{m-1}h \end{bmatrix}^T ,$$

Moreover

$$\Omega(t) = \begin{bmatrix} a_0 & a_1 & \cdots & a_{m-1} & c_0 & c_1 & \cdots & c_{m=1} \\ Wa_{m-1} & Wa_0 & \cdots & Wa_{m-2} & Wc_{m=1} & Wc_0 & \cdots & Wc_{m-2} \\ W^2a_{m-2} & W^2a_{m-1} & \cdots & W^2a_{m-3} & W^2c_{m-2} & W^2c_{m=1} & \cdots & W^2c_{m-3} \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ W^{m-1}a_1 & W^{m-1}a_2 & \cdots & W^{m-1}a_0 & W^{m-1}c_1 & W^{m-1}c_2 & \cdots & W^{m-1}c_0 \\ c_0 & c_1 & \cdots & c_{m-1} & a_1 & a_2 & \cdots & a_{m-1} \\ Wc_{m-1} & Wc_0 & \cdots & Wc_{m-2} & Wa_{m=1} & Wa_0 & \cdots & Wa_{m-2} \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ W^{m-1}c_1 & W^{m-1}c_2 & \cdots & W^{m-1}c_0 & W^{m-1}a_1 & W^{m-1}a_2 & \cdots & W^{m-1}a_0 \end{bmatrix}$$

is a matrix of functions from the space $H_\Gamma(\omega)$ corresponding to the operator $\overline{\Omega}$.

Then the system (5.1)-(5.6) can be rewritten as the form:

$$\overline{\Omega}H + MH = G , \quad H \in E \tag{5.8}$$

Where

$$G = \begin{bmatrix} g & Wg & ... & W^{m-1}g & Sg & WSg & ... & W^{m-1}Sg \end{bmatrix}^T ,$$

and M is a diagonal matrix its diagonal numbers take the form

$$dig\, M = \{N, WNW^{m-1}, ..., W^{m-1}NW, N_1S, N_2SW^{m-1}, ..., N_mSW\}$$

Theorem 5.1 Assume that

$$\det \Omega(t) \neq 0 \qquad \forall t \in \Gamma, \tag{5.9}$$

$$\left\| \overline{\Omega}^{-1} M \right\|_E < 1. \tag{5.10}$$

Then the operator $T'(u_0)$ is invertible, moreover

$$\left\| (T'(u_0))^{-1} \right\|_{H_\Gamma} \leq \frac{\left\| \Omega^* \right\|_{H_\Gamma}}{1 - \left\| \Omega^{-1} M \right\|_E} \left(\frac{1}{n} + H_\Gamma^\omega(\Omega_0) \right), \tag{5.11}$$

where

$$n = \min_{t \in \Gamma} |\det \Omega(t)|, \qquad \Omega_0(t) = \frac{1}{\det \Omega(t)},$$

and Ω^* be the adjoint matrix of Ω.

Proof.

It is well know that the condition (5.9) is necessary and sufficient condition for the invertibility of the operator $\overline{\Omega}$ on E, moreover the equation (5.8) is equivalent to the following equation

$$H = \overline{\Omega}^{-1}G - \overline{\Omega}^{-1}MH, \quad H \in E.$$

The problem of the invertibility of the operator $\overline{\Omega} + M$ can be reduced to the following fixed-point problem

$$H = PH, \ PH = \overline{\Omega}^{-1}G - \overline{\Omega}^{-1}MH, \ H \in E,$$

where

$$\| PH_1 - PH_2 \|_{H_\Gamma} \leq \left\| \overline{\Omega}^{-1} M \right\|_E \| H_1 - H_2 \|_{H_\Gamma} ,$$

From condition (5.10) and the contraction theorem, it follows that for every $G \in E$, the operator P has a unique fixed point. Then the operator $\overline{\Omega} + M$ (and therefore $T'(u_0)$) is invertible and

$$\| (T'(u_0))^{-1} \|_{H_\Gamma} \leq \frac{\left\| \overline{\Omega}^{-1} \right\|_{H_\Gamma}}{1 - \left\| \overline{\Omega}^{-1} M \right\|_E}$$

moreover

$$\left\| (T'(u_0))^{-1} \right\|_{H_\Gamma} \leq \frac{\left\| \Omega^* \right\|_{H_\Gamma}}{1 - \left\| \Omega^{-1} M \right\|_E} \left(\frac{1}{n} + H_\Gamma^\omega(\Omega_0) \right).$$

Assume that

$$b = \frac{\left\|\Omega^{*}\right\|_{H_\Gamma}}{1-\left\|\Omega^{-1}M\right\|_{E}}\left(\frac{1}{n}+H_\Gamma^{\omega}(\Omega_0)\right)$$

and $a = b(\left\|B_1 u_0\right\|_{H_\Gamma} + \left\|\Lambda_\Psi u_0\right\|_{H_\Gamma})$

In fact, we have

$$Q(u) = Q(u_0) + Q(u) - Q(u_0);\ Q(u) = T'(u)$$

implies

$$Q(u)^{-1} = \left[I + Q(u_0)^{-1}(Q(u) - Q(u_0))\right]^{-1} Q(u_0)^{-1},$$

Consequently,

$$\left\|Q(u)^{-1}\right\|_{H_\Gamma} \le \frac{\left\|Q(u_0)^{-1}\right\|_{H_\Gamma}}{1-\left\|Q(u_0)^{-1}\right\|_{H_\Gamma}\left\|Q(u)-Q(u_0)\right\|_{H_\Gamma}}.$$

Hence,

$$\left\|Q(u)^{-1}\right\|_{H_\Gamma} \le \frac{b}{1-b\theta(r)},\quad (u \in B(u_0, r)).$$

Where

$$\theta(r) = \sup\left\{\left\|L(u)-L(u_0)\right\|_{H_\Gamma} :\ \left\|u-u_o\right\|_{H_\Gamma} \le r\right\},$$

$$\theta : [0,\infty) \to [0,\infty)$$

Such that $0 \le \theta(r) \le \mu(r),\ (0 \le r \le R)$.

Therefore, the following theorems are valid.

Theorem 5.2 Suppose that the function (2.9) has a unique zero $r_* \in [0, R]$ and that $\phi(R) \le 0$. Then equation (1.1) has a solution $u_* \in B(u_0, r_*)$ this solution is unique in the ball $B(u_0, R)$.

Theorem 5.3 Under the hypotheses of Lemma 2.1 the approximation (2.2) are defined for all n belong to the ball $B(u_0, q_*)$, are converging to a solution u_* of (1.1) and satisfy the estimates (2.12) and (2.13).

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Received: Dec 23, 2017; Accepted: Jan 20, 2018